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Disordered wires from a geometric viewpoint

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Abstract. The Mello-Pereyra-Kumar theory is extended to arbitrary spin. Classification yields three universality classes. The average over disorder is related to geometrical concepts. The solution of the resulting differential equation is discussed, localization lengths and universal conductance fluctuations are computed.

1. Introduction

Some years ago, the experimental discovery of conductance fluctuations at low temperatures of systems that are small with respect to the inelastic scattering length initiated several theoretical studies. Since many results do not depend on details of the microscopic models, Mello, Pereyra and Kumar presented a theory dealing with the statistical distribution of the transfer matrix of a macroscopic quasi-one-dimensional conductor [1]. They discussed several features of their model in the case of a time-reversal-invariant system. In a subsequent publication Mello presented an interesting calculation yielding the universal variance of the fluctuating conductance [2].

The present paper has two purposes. First we intend to describe the model of Mello *et al* from a global, coordinate-independent point of view. This leads to the well known mathematical task of solving the heat conduction problem on certain non-compact Riemannian manifolds. As a consequence, the solution of the basic differential equation, which is equivalent to the Focker-Planck equation in Mello's setup, is identified as a heat kernel. It may be described in a semi-explicit manner by some geometrical considerations. The second aim is to extend the model to the more general situation where either time reversal symmetry is broken or the spin of the particles, which we allow to be arbitrary, participates in the interaction. Classification yields three universality classes. As a physical application the dependence of the localization length on the number of scattering channels is discussed. Finally, Mello's calculation of the universal variance of conductance fluctuations is carried over to the more general case.

2. Physical situation, symmetries and classification

The system we are dealing with is a quasi-one-dimensional conductor with static disorder. We assume it to be long enough for its physical properties to be dominated by a single length scale. This implies natural units, and the spatial extension will be described by a dimensionless parameter $s \in \mathbb{R}^+$. To have a non-empty range of validity, temperature has to be low enough to give an inelastic scattering length large compared

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to unity. We always think of our sample as being connected to two half-infinite ideal conductors. This reflects the experimental setup and lends itself well to a scattering-theoretic approach. Finally we endow the particles with a spin degree of freedom that transforms according to an irreducible unitary representation of SU(2).

To discuss the symmetries of the problem, we consider an arbitrary but fixed sample. Its physical properties are determined by a scattering matrix $S \in \mathrm{U}(2n)$, $n \in \mathbb{N}$ counting the number of scattering channels. S maps incoming states $\psi_{\mathrm{in}}^{\mathrm{left}}, \psi_{\mathrm{in}}^{\mathrm{right}} \in \mathbb{C}^n$ onto outgoing ones $\psi_{\mathrm{out}}^{\mathrm{left}}, \psi_{\mathrm{out}}^{\mathrm{right}} \in \mathbb{C}^n$ via

$$\psi_{\mathrm{out}} = \begin{pmatrix} \psi_{\mathrm{out}}^{\mathrm{left}} \\ \psi_{\mathrm{out}}^{\mathrm{right}} \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{in}}^{\mathrm{left}} \\ \psi_{\mathrm{in}}^{\mathrm{right}} \end{pmatrix} = S\psi_{\mathrm{in}}. \tag{1}$$

This relation may be reformulated in terms of a 'transfer matrix' $T \in \mathrm{U}(n, n)$ by

$$\psi^{\mathrm{right}} = \begin{pmatrix} \psi_{\mathrm{out}}^{\mathrm{right}} \\ \psi_{\mathrm{in}}^{\mathrm{right}} \end{pmatrix} = \begin{pmatrix} Z_0' & Z_1 \\ Z_1' & Z_0 \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{in}}^{\mathrm{left}} \\ \psi_{\mathrm{out}}^{\mathrm{left}} \end{pmatrix} = T\psi^{\mathrm{left}}. \tag{2}$$

The physical operation of connecting two samples amounts to multiplication of their transfer matrices.

The implementation of a possible time reversal symmetry involves an antiunitary operator represented by

$$\psi \in \mathbb{C}^n \to \tau\bar{\psi} \in \mathbb{C}^n \qquad \tau \in \mathrm{U}(n). \tag{3}$$

Let the spin equal $J \in \mathbb{N}/2$, and let $m \in \mathbb{N}$ be the number of channels so that $n = m(2J+1)$. Adopting the usual conventions, we write τ as

$$\tau = 1^{(m)} \otimes \exp(\mathrm{i}\pi J_2) \tag{4}$$

J_2 being the second component of angular momentum which we choose to be purely imaginary, as in the standard representation [3]. Consequently $\tau = \bar{\tau} = \varepsilon\tau^{\dagger} = \varepsilon\tau^{-1} = \varepsilon\tau^{\mathrm{t}}$, $\varepsilon = -(-1)^{\dim[J]}$. Especially $\tau^2 = \varepsilon \times 1^{(n)}$.

We have an induced transformation on $\mathrm{U}(n, n)$ also denoted by τ:

$$\begin{aligned} &\tau : \mathrm{U}(n, n) \to \mathrm{U}(n, n) \\ &\qquad g \to \tau(g) = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \bar{g} \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}^{-1}. \end{aligned} \tag{5}$$

τ acting on $\mathrm{U}(n, n)$ is an involution, $\tau^2 = 1^{(n)}$, and respects the group multiplication:

$$\tau(g_1 g_2) = \tau(g_1)\tau(g_2). \tag{6}$$

Thus it fixes a subgroup $\mathrm{Z}_2 \times \mathrm{G}_\varepsilon \subset \mathrm{U}(n, n) \simeq \mathrm{U}(1) \times \mathrm{SU}(n, n)$ which essentially depends only on the value of $\varepsilon \in \{-1, 1\}$.

We use the representation of the Lie algebra [4]

$$\mathrm{su}(n, n) = \left\{ \begin{pmatrix} U & X \\ \overline{X^{\mathrm{t}}} & V \end{pmatrix} \in M_{2n}(\mathbb{C}) \,|\, U, V \in \mathrm{u}(n), \mathrm{tr}(U+V) = 0, X \in M_n(\mathbb{C}) \right\} \tag{7}$$

and conjugate by

$$X \to \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}^{-1} X \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \qquad X \in \mathrm{su}(n, n). \tag{8}$$

For the Lie algebras g_ε of the conjugated G_ε, one then finds

$$g_\varepsilon = \left\{ \begin{pmatrix} U & Z \\ \overline{Z'} & \overline{U} \end{pmatrix} \in M_{2n}(\mathbb{C}) \mid U \in u(n), Z \in M_n(\mathbb{C}), Z = \varepsilon Z^t \right\}. \tag{9}$$

A further conjugation by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1^{(n)} & -i1^{(n)} \\ -i1^{(n)} & 1^{(n)} \end{pmatrix} \tag{10}$$

establishes the algebra isomorphisms

$$\begin{aligned} g_1 \simeq & \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^t \end{pmatrix} \in M_{2n}(\mathbb{R}) \mid X_i \in M_n(\mathbb{R}), i = 1, 2, 3, X_j = X_j^t, j = 2, 3 \right\} \\ & = \mathrm{sp}(n, \mathbb{R}) \end{aligned} \tag{11}$$

and

$$\begin{aligned} g_{-1} \simeq & \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z_2} & \overline{Z_1} \end{pmatrix} \in M_{2n}(\mathbb{C}) \mid Z_i \in M_n(\mathbb{C}), i = 1, 2, Z_1 = -Z_1^t, Z_2 = Z_2^t \right\} \\ & = \mathrm{so}^*(2n) \end{aligned} \tag{12}$$

(compare with [4]). We finally summarize the three universality classes:

<u>Time reversal</u>	<u>Transfer matrix group</u>	
none	$U(n, n) \simeq U(1) \times SU(n, n)$	
integer spin	$Z_2 \times Sp(n, \mathbb{R})$	(13)
half-integer spin	$Z_2 \times SO^*(2n)$.	

Identification of a real physical system within this framework depends on the way in which the spin participates in the interaction with the background. The first case applies when magnetic interactions are present. The second class contains time-reversal-invariant samples without essential spin-background interaction. This is the situation treated by Mello and coworkers [1]. The last group refers to electronic systems with spin-orbit scattering present.

The appearance of three universality classes of models is a common feature of several statistical theories. It was found by Dyson in his studies of distributions of energy levels [5], and related classifications apply to disordered electron systems, as was pointed out by Wegner [6] and Efetov [7].

For notational convenience let us agree that G_0' denotes one of above symmetry groups and G_0 its semisimple part in the sense indicated in the table. The model turns out to be sensitive to G_0 only.

Finally let us recall that 'the' observable of our problem, the Landauer conductance $\mathcal{G}$ [8], can be expressed in terms of the transmission blocks of the S-matrix as follows:

$$\mathcal{G} = \tfrac{1}{2}[\mathrm{tr}(t^+ t) + \mathrm{tr}(t'^+ t')]. \tag{14}$$

With the abbreviation $Z = Z_1 Z_0^{-1}$ $(Z' = Z_1' Z_0'^{-1})$ this reduces to an expression in terms of the transfer matrix T as

$$\mathcal{G} = \mathrm{tr}(1^{(n)} - Z^+ Z) = \mathrm{tr}(1^{(n)} - Z'^+ Z'). \tag{15}$$

3. Concept of the theory

Having introduced the basic ingredients in the last section, we are now able to define the mathematical counterpart of our physical setup. For two reasons we will do that from a global point of view independent of a special choice of coordinates. The first is simply to present an alternative approach to be contrasted with Mello's maximum-entropy principle [1]. The separation into structure on the one hand, and special features that result from a particular choice of parameters on the other, becomes transparent and allows an *a priori* glance at properties and drawbacks of the model. The second is that the language chosen seems to be well suited in the sense that the solution of the theory can be expressed in terms of a well known mathematical concept (the heat kernel) and is immediately available in a semi-explicit form.

A suitable specification of the configuration space of the problem is achieved by observing that a specific sample of length $s \in \mathbb{R}^+$ and transfer matrix $T(s) \in \mathrm{G}_0$ may possess a variety of possible internal structures. This could in principle be analysed by successively growing the sample and thus tracing a path $T \in C^0([0, s], G_0)$ starting at the identity e and ending at the transfer matrix $T(s)$. Consequently a plausible choice of configuration space is

$$\Omega^s_{\mathrm{G}_0} = \{\gamma \in C^0([0, s], \mathrm{G}_0) | \gamma(0) = e\}. \tag{16}$$

At this point one is tempted to try and proceed by appealing to the theory of Brownian motion, the clear paradigm of a probability theory on some path space. However, its formulation relies in an essential way on the *Riemannian* structure of the underlying manifold. In this context it is important to realize that the non-compact semisimple Lie groups are only *pseudo Riemannian* when endowed with their natural left-invariant geometry. This difficulty could, of course, be cured by introducing some other geometry, different from the natural one, but we have no way of deciding what geometry that should be and the theory would become dependent on new and arbitrary input.

A sensible compromise is suggested by the following observations. (i) Let K_0 denote a maximal compact subgroup of G_0. Explicit construction of K_0, and use of the relations given at the end of the previous section, show that the Landauer expression for the conductance $\mathscr{G}$ is bi-invariant under K_0. (ii) Each of the groups G_0 brings with it a Riemannian manifold $\mathrm{G}_0/\mathrm{K}_0$ obtained by forming left cosets with respect to K_0. Consequently, if we decide to answer only a restricted set of questions, we may project on $\mathrm{G}_0/\mathrm{K}_0$. It is fortunate that this projection matches perfectly with the invariance properties of the Landauer conductance $\mathscr{G}$, as indicated.

For the time being we only intend to study properties of the sample as a whole and consequently end up with the following concept.

We postulate the average over disorder to correspond to Brownian motion on $\mathrm{G}_0/\mathrm{K}_0$. Any physical sample is represented by some random path on $\mathrm{G}_0/\mathrm{K}_0$ connecting the origin $e\mathrm{K}_0$ to $T(s)\mathrm{K}_0$. $T(s)$ is the transfer matrix for the sample with length $s \in \mathbb{R}^+$ in question. Let $\mathrm{d}g_{\mathrm{K}_0}$ be the invariant volume form on $\mathrm{G}_0/\mathrm{K}_0$ and $\omega_s = p_s \, \mathrm{d}g_{\mathrm{K}_0}$, $p_s : \mathrm{G}_0/\mathrm{K}_0 \to \mathbb{R}^+$, the induced probability density. The assumption of Brownian motion implies that p_s satisfies the heat equation

$$\partial_s p_s = \tfrac{1}{2}\Delta p_s \qquad \lim_{s\to 0} p_s = \delta_0. \tag{17}$$

Δ denotes the Laplace–Beltrami operator on $\mathrm{G}_0/\mathrm{K}_0$ and δ_0 is Dirac's distribution with respect to the origin $o = e\mathrm{K}_0 = \mathrm{K}_0$. The prefactor is a matter of convention via redefinition of length scale. As observables we choose K_0-invariant functions $f : \mathrm{G}_0/\mathrm{K}_0 \to \mathbb{R}$ integrable with respect to ω_s. The expectation value of f will be written as

$$\langle f \rangle_s = \int_{\mathrm{G}_0/\mathrm{K}_0} f\omega_s. \tag{18}$$

We finish with some remarks. First, let $\tilde{}$ be the operation of averaging over K_0 in the sense of

$$\tilde{f}(x) = \frac{1}{\mathrm{vol}(\mathrm{K}_0)} \int_{\mathrm{K}_0} f(kx)\, \mathrm{d}k \qquad x \in \mathrm{G}_0/\mathrm{K}_0 \tag{19}$$

$\mathrm{d}k$ being the invariant volume form on K_0. By the symmetries at hand, namely $p_s(kx) = p_s(x)$ for $k \in \mathrm{K}_0$, we have

$$\langle f \rangle_s = \langle \tilde{f} \rangle_s \tag{20}$$

for any suitable $f : \mathrm{G}_0/\mathrm{K}_0 \to \mathbb{R}$. Consequently the model does not distinguish left cosets. An adequate projection of observables defined on the whole group G_0 is given by

$$f(x) \to \frac{1}{\mathrm{vol}(\mathrm{K}_0)^2} \int_{\mathrm{K}_0} \int_{\mathrm{K}_0} f(k_1 x k_2)\, \mathrm{d}k_1\, \mathrm{d}k_2 \qquad x \in \mathrm{G}_0. \tag{21}$$

We emphasize that from our point of view this construction does not necessarily imply that the 'phases' K_0 are equally distributed with respect to $\mathrm{d}k$. Questions concerning the K_0-degrees of freedom are simply not allowed, as explained above. Second we remark that Mello's Focker–Planck equation [1] is the just given heat equation represented on G_0 with a special choice of coordinates.

A further satisfying test of consistency is the observation that it does not matter whether one passes to the quotient before demanding some time-reversal invariance or first restricts to the subgroups and then passes to the quotient, since the involution τ descends to a well defined map on $\mathrm{G}_0/\mathrm{K}_0$ that fixes the coset spaces belonging to the time-reversal-invariant systems.

4. Some conventions and basic objects

We try to choose our notational conventions in a standard manner, so that most of the symbols should be clear from the context. To begin with we list some facts useful in the discussion of the model. Let $\mathrm{M} = \mathrm{G}/\mathrm{U}$, G being a semisimple non-compact Lie group, $\mathrm{U} \subset \mathrm{G}$ a maximal compact subgroup. G acts as a transformation group on M by $h(g\mathrm{U}) = hg\mathrm{U}$. We denote the origin by $o = e\mathrm{U}$. The Riemannian structure on M is induced via restriction of the Cartan–Killing form to $T_0(\mathrm{G}/\mathrm{U}) \simeq p$, $g = u \oplus p$. We denote with Exp the exponential map at the origin o:

$$\begin{aligned} \mathrm{Exp} : T_o(\mathrm{G}/\mathrm{U}) &\to \mathrm{G}/\mathrm{U} \\ X &\to \mathrm{Exp}(X) = \exp(X)\mathrm{U}. \end{aligned} \tag{22}$$

This yields a global chart. The invariant volume form $\mathrm{d}g_{\mathrm{U}}$ pulls back to [4]

$$(\mathrm{Exp}^* \, \mathrm{d}g_{\mathrm{U}})_X = \det\left(\left. \frac{\sinh(\mathrm{ad}\, X)}{\mathrm{ad}\, X} \right|_p \right) (\mathrm{d}g_{\mathrm{U}})_o. \tag{23}$$

By the action of $\mathrm{Ad}(\mathrm{U})$, p may be reduced to a maximal Abelian subspace $a \subset p$, $p = \mathrm{Ad}(\mathrm{U})a$. Calculation of the above determinant requires knowledge of the eigenvalues of $(\mathrm{ad}\, H)^2 : p \to p$, $H \in a$. These give rise to a set $\Delta^+ \in a^*$ of linear functionals that are positive when restricted to the 'positive Weyl chamber' $a^+ \subset a$, which is bounded by subsets of hyperplanes $\alpha = 0$, $\alpha \in \Delta^+$. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} m_\alpha \alpha$, $\delta = \prod_{\alpha \in \Delta^+} (\sinh \circ \alpha)^{m_\alpha}$, $\pi = \prod_{\alpha \in \Delta^+} \alpha^{m_\alpha}$ where m_α gives the multiplicity of $\alpha \in \Delta^+$. In the case of complex G all m_α are equal to 2, which is a consequence of the fact that all root subspaces of a complex Lie algebra have complex dimension 1.

In the application we have in mind, G will play the role of complexified transfer matrix group G_0. On the space G_0/K_0 the analogous constructions will be denoted by the same letters with subscript 0 whenever no confusion is possible.

In the representation of the Lie algebras as given in the second section, the maximal Abelian subspaces consist of matrices of the type

$$\begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} \qquad H = \mathrm{diag}(h_1, \ldots, h_n) \in \mathrm{M}_n(\mathbb{R}) \tag{24}$$

in the cases $\mathrm{su}(n, n)$ and $\mathrm{sp}(n, \mathbb{R})$ and

$$H = \mathrm{diag}\left(\begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & h_{n/2} \\ -h_{n/2} & 0 \end{pmatrix}\right) \in \mathrm{M}_n(\mathbb{R}) \tag{25}$$

in the case $\mathrm{so}^*(2n)$. (For our purpose n is always even in the last case.) In all cases, let $e_i(H) = h_i$. We give a list of positive roots and multiplicities for completeness:

	Roots		
Algebra	$2e_i$	$e_i + e_j,\ i > j$	$e_i - e_j,\ i > j$
$\mathrm{su}(n, n)$	1	2	2
$\mathrm{sp}(n, \mathbb{R})$	1	1	1
$\mathrm{so}^*(2n)$	1	4	4.

(26)

5. The probability density

Calculation of the probability density p is, by construction, equivalent to determining the heat kernel on a non-compact Riemannian symmetric space G_0/K_0, G_0 a non-compact semisimple real Lie group, K_0 a maximal compact subgroup. This problem is well known in mathematics. The work of Flensted-Jensen [9] reduces it to the case when G_0 has a complex structure. The heat kernel in this case is expressible in terms of elementary functions. To make this discussion essentially self-contained we give an independent argument in terms of more elementary geometric notions and refer to Flensted-Jensen for a more general background on Fourier analysis. While prepraring this publication, we became aware of the work of Anker and Lahoue [10] where the same argument is applied.

We start by remembering two general manipulations. Let $n_0 = \dim(G_0/K_0)$ and let $|x|$ denote the geodesic distance of a point $x \in G_0/K_0$ from the origin. To split off the correct singularity as $s \to 0$ let $p = \mathscr{E} \times q$ with

$$\mathscr{E} : R^+ \times G_0/K_0 \to R^+$$

$$(s, x) \to \mathscr{E}_s(x) = \frac{1}{(2\pi s)^{n_0/2}} \exp\left(-\frac{|x|^2}{2s}\right). \tag{27}$$

This leads to

$$\partial_s q = \tfrac{1}{2}\Delta q - A(\ln(\varphi^{1/2} q))q \qquad \lim_{s\to 0} q(o) = 1. \tag{28}$$

A denotes the vector field (viewed as a differential operator) specified by

$$A = \mathrm{grad}\left(\frac{|\cdot|^2}{2s}\right). \tag{29}$$

$|\cdot|$ is the function $x \to |x|$ and $\varphi : G_0/K_0 \to \mathbb{R}$ is given by

$$(\varphi \circ \mathrm{Exp})(X) = \det\left(\frac{\sinh \mathrm{ad}\, X}{\mathrm{ad}\, X}\bigg|_{p_0}\right) \tag{30}$$

and measures the expansion of volume with respect to the tangent space at the origin.

A further substitution $q = \varphi^{1/2} v$ yields

$$\partial_s v = \tfrac{1}{2}\Delta v + Vv - B(v) \qquad V = \tfrac{1}{2}\varphi^{1/2}(\Delta \varphi^{-1/2}) \qquad B = \mathrm{grad}\left(\ln \varphi^{1/2} + \frac{|\cdot|^2}{2s}\right). \tag{31}$$

Examination on the maximal Abelian subspace leads to

$$\lim_{x\to\infty} V(x) = -\tfrac{1}{2}\|\rho_0\|^2 \tag{32}$$

$$p_s(x) \sim \frac{1}{(2\pi s)^{n_0/2}} \varphi^{1/2}(x) \exp\left(-\frac{|x|^2}{2s}\right) \exp\left(-\frac{\|\rho_0\|^2}{2} s\right) \tag{33}$$

as $x \to \infty$. This expression is simple enough to permit a discussion of localization lengths.

When G_0 additionally possesses a complex structure, the root multiplicities all equal 2, implying as a matter of fact $V = -\frac{1}{2}\|\rho\|^2$. This turns the above formula into an identity.

Now let G be the complexification of G_0, U a maximal compact subgroup containing K_0 and P the heat kernel on G/U. We now view G_0/K_0 as a totally geodesic submanifold on G/U. The tangent space $T_0 G/U \simeq \mathrm{i}k_0 \oplus p_0$ decomposes transversally. The negative sectional curvature of the spaces at hand ensures that the map

$$\begin{aligned} \phi &: G_0/K_0 \times \mathrm{i}k_0 \to G/U \\ &(x, K) \to \phi(x, K) = \exp(K)x \end{aligned} \tag{34}$$

is a diffeomorphism onto G/U. Thus G/U decomposes into G_0/K_0 and additional transverse degrees of freedom. The important fact is that integration along $\mathrm{i}k_0$ relates the heat equations on both spaces as can be seen as follows [10]. A straightforward computation yields ($x = \mathrm{Exp}(X)$)

$$\begin{aligned} (\phi^* \, \mathrm{d}g_U)_{(x,K)} &= \det\begin{pmatrix} \mathrm{i}d & 0 \\ * & \cosh \mathrm{ad}\, X(\sinh \mathrm{ad}\, K/\mathrm{ad}\, K) \end{pmatrix} (\mathrm{d}g_{K_0})_x (\mathrm{d}k_{K_0})_0 \\ &= \det(\cosh \mathrm{ad}\, X|_{\mathrm{i}k_0})(\mathrm{d}g_{K_0})_x (\mathrm{Exp}^* \, \mathrm{d}k_{K_0})_K . \end{aligned} \tag{35}$$

Inspection of the Iwasawa decomposition [4] shows

$$\det(\cosh \mathrm{ad}\, X|_{\mathrm{i}k_0}) = \det(\cosh \mathrm{ad}\, X|_{p_0}) \tag{36}$$

by suitable orientation of $\mathrm{i}k_0$.

With $\eta : G_0/K_0 \to G_0/K_0$ given by $\eta(\mathrm{Exp}(Y)) = \mathrm{Exp}(2Y)$, we finally have

$$(\phi^* \mathrm{d}g_U)_{(x,K)} = 2^{-n_0} (\eta^* \mathrm{d}g_{K_0})_x (\mathrm{Exp}^* \, \mathrm{d}k_{K_0})_K . \tag{37}$$

Let $f : G/U \to \mathbb{R}$ be a function such that

$$(\hat{f} \circ \eta)(x) = \frac{1}{2^{n_0}} \int_{ik_0} (f \circ \phi)(x, K)(\mathrm{Exp}^* \, \mathrm{d}k_{K_0}) \tag{38}$$

exists. Additional U—invariance of f implies K_0—invariance of $\hat{f}$ and

$$(\hat{f} \circ \eta)(x) = \frac{1}{2^{n_0}} \frac{1}{\mathrm{vol}(K_0)} \int_K f(kx) \, \mathrm{d}k \tag{39}$$

K denoting complexified K_0 as a subgroup of G. The K-radial part [11] Δ' of the Laplace–Beltrami operator on G/U is thus related to the radial part Δ_0' of the Laplace–Beltrami operator Δ_0 on G_0/K_0:

$$\Delta' = 4(\Delta_0')^{\eta}. \tag{40}$$

The solution of the heat equation (17) may be expressed as

$$p_s(x) = \frac{1}{2^{n_0}} \frac{1}{\mathrm{vol}(K_0)} \int_K P_{s/4}(k\eta^{-1}(x)) \, \mathrm{d}k. \tag{41}$$

P itself is given by a formula of type (33).

Looking at G_0/K_0 with its geometry induced by the Killing form on $g_0 \times g_0$ requires a rescaling of the metric by a factor of one half.

Although it reflects the large amount of symmetry of the problem at hand, the final formula is still rather difficult to handle. In particular, it is not obvious how a sharp upper bound may be obtained. For a different approach yielding an interesting inequality in the $G_0 = SU(n, n)$ case, see Anker [12], who in addition gives a conjecture on a sensitive upper bound for the heat kernel on Riemannian symmetric spaces.

In the single-channel case the above formula is an effective means of giving the distribution function of conductance explicitly by simple manipulations avoiding the use of special features of Legendre functions. Since it was not given in [13], we briefly outline this case which essentially is the textbook example of heat conduction on the hyperbolic plane [9, 14].

We have

$$\begin{aligned} G_0/K_0 &= SU(1,1)/U(1) \simeq SO(2,1)/SO(2) \\ G/U &= SL(2,\mathbb{C})/SU(2) \simeq SO(3,1)/SO(3). \end{aligned} \tag{42}$$

As a model we use

$$G/U \simeq \{(x^0, \boldsymbol{x}) \in \mathbb{R}^+ \times \mathbb{R}^3 | (x^0)^2 - |\boldsymbol{x}|^2 = -1\} \tag{43}$$

with SO(3, 1)-invariant geometry via restriction of $\mathrm{d}s^2 = |\mathrm{d}\boldsymbol{x}|^2 - (\mathrm{d}x^0)^2$. With respect to polar coordinates by $\boldsymbol{x} = \sinh(t)\boldsymbol{\xi}$, $t \in \mathbb{R}^+$, $\boldsymbol{\xi} \in S^2$, we have

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \sinh^2(t)|\mathrm{d}\boldsymbol{\xi}|^2 \tag{44}$$

and

$$P_s(t) = \frac{1}{(2\pi s)^{3/2}} \frac{t}{\sinh(t)} \exp\left(-\frac{t^2}{2s}\right) \exp\left(-\frac{s}{2}\right). \tag{45}$$

Imbed G_0/K_0 by $x^1=0$, $x=(x^1,x^2,x^3)$; put $x=(0,\sinh(\sigma)e)$, $\sigma\in\mathbb{R}^+$, $e\in S^1$ and generate G/U by mapping

$$(\sigma, e, u)\in\mathbb{R}^+\times S^1\times\mathbb{R}\to\begin{pmatrix}\cosh(u) & \sinh(u) & 0 & 0\\ \sinh(u) & \cosh(u) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cosh(\sigma)\\ 0\\ \sinh(\sigma)e\end{pmatrix}. \tag{46}$$

Comparing x^0-components yields $t(u,\sigma)=\cosh^{-1}(\cosh(u)\cosh(\sigma))$ and finally

$$\begin{aligned} p_s(\sigma) &= \frac{1}{4}\int_{-\infty}^{\infty} P_{s/4}\left(t\left(u,\frac{\sigma}{2}\right)\right)\,\mathrm{d}u \\ &= \frac{\sqrt{2}}{(2\pi s)^{3/2}}\,\mathrm{e}^{-s/8}\int_{\sigma}^{\infty}\frac{x\,\mathrm{e}^{-x^2/2s}}{\sqrt{\cosh(x)-\cosh(\sigma)}}\,\mathrm{d}x \end{aligned} \tag{47}$$

by substitution.

Finally, with respect to above coordinates $\mathscr{G}=\cosh(\sigma/2)^{-2}$ and for the conductance distribution function we have:

$$w_s(\mathscr{G})=\frac{\pi}{(2\pi s)^{3/2}}\frac{1}{\mathscr{G}^2}\,\mathrm{e}^{-s/8}\int_0^{\infty}\frac{\cosh^{-1}(\cosh(u)/\sqrt{\mathscr{G}})}{\sqrt{\cosh(u)^2-\mathscr{G}}}\exp\left(-\frac{\cosh^{-1}(\cosh(u)/\sqrt{\mathscr{G}})^2}{s}\right)\mathrm{d}u \tag{48}$$

while expectation values are computed as

$$\langle f\rangle_s=\int_0^1 f(\mathscr{G})w_s(\mathscr{G})\,\mathrm{d}\mathscr{G}. \tag{49}$$

6. Localization lengths

This and the following section are devoted to the two main physical features of the model. Let us first deal with the localization phenomenon. Put

$$\mathscr{L}=\{l\in\mathbb{R}^+|\mathrm{e}^{s/l}\langle\mathscr{G}\rangle_s \text{ is bounded above as } s\to\infty\} \tag{50}$$

and define the localization length s_0 to be the greatest lower bound of $\mathscr{L}$ ($s_0=\infty$ if $\mathscr{L}=\varnothing$). Since this bound depends on the convergence properties of the integral $\langle\mathscr{G}\rangle_s$ at spatial infinity, we use the approximate solution of the heat equation. We have to study the integral

$$\begin{aligned} &\int_{G_0/K_0}\mathscr{G}(x)p_s(x)\,\mathrm{d}g_{K_0} \\ &\quad=\int_{a_0^+}(\mathscr{G}\circ\mathrm{Exp})(H)\frac{1}{(2\pi s)^{n_0/2}}\exp(-\|H\|^2/2s) \\ &\qquad\times\exp(-\|\rho\|^2 s/2)\pi_0^{1/2}(H)\delta_0^{1/2}(H)\,\mathrm{d}^r H. \end{aligned} \tag{51}$$

Referring to the choice of a_0 given at the end of section 4

$$(\mathscr{G}\circ\mathrm{Exp})(H)\sim\sum_{i=1}^{n}4\exp(-2e_i(H)) \tag{52}$$

within the first two classes and

$$(\mathscr{G}\circ \mathrm{Exp})(H) \sim \sum_{i=1}^{n/2} 8\exp(-2e_i(H)) \tag{53}$$

in the time-reversal-invariant half-integer-spin case. Up to (inverse) powers of s the relevant information is contained in integrals of the type

$$I_s(\beta) = \frac{1}{(2\pi s)^{r/2}} \int_{a_0^+} \exp(-\beta(H)) \exp(-\|H\|^2/2s) \exp(-\|\rho_0\|^2 s/2) \exp(\rho_0(H))\, \mathrm{d}^r H \tag{54}$$

where β is a positive root. The main clue is that for $c = \langle \beta, \rho \rangle / \|\beta\|^2 > 0$ we have $\beta(H_{\rho_0} - cH_\beta) = 0$. Thus $H_{\rho_0} - cH_\beta \in \partial a_0^+$ and by convexity $H_{\rho_0} - \sigma H_\beta \in a_0^+$ for $0 \leqslant \sigma < c$. It is useful to reexpress

$$I_s(\beta) = \frac{\exp(-\|\rho_0\|^2 s/2)}{\pi^{r/2}} \int_{a_0^+ - \sqrt{s/2}(H_{\rho_0} - cH_\beta)} \exp[-\sqrt{2s}(1-c)\beta(\xi)] \times \exp(-\|\xi\|^2) \exp[(s/2)\|\rho_0 - c\beta\|^2]\, \mathrm{d}^r\xi. \tag{55}$$

One has to distinguish the cases $c<1$, $c=1$ and $c>1$. The important one turns out to be $c<1$, β being simple. In this case the region $a_0^+ - \sqrt{s/2}(H_{\rho_0} - cH_\beta)$ eventually fills the half space $\beta \geqslant 0$ as $s \to \infty$ and

$$I_s(\beta) \sim \frac{\exp(-c^2\|\beta\|^2 s/2)}{\sqrt{2\pi s}\,(1-c)\|\beta\|}. \tag{56}$$

Up to a common, n-independent factor depending on normalization of the metric we arrive at the following table:

Time reversal	s_0
none	$\sim 8n$
integer spin	$\sim 4(n+1)$
half-integer spin	$\sim 8(n-1)$.

(57)

As expected, the localization length scales with the number of scattering channels in all cases.

7. Universal conductance fluctuations

Perhaps the most interesting feature of the model is the appearance of universal conductance fluctuations in the asymptotic $n \to \infty$ limit, as demonstrated by Mello [2]. We adopt his computational method to obtain a list of all three universality classes. Mello's basic idea was to study the expectation $\langle \mathscr{G}^p \rangle_s$ by assuming asymptotics with respect to n as

$$\langle \mathscr{G}^p \rangle_s = n^p f_{p,0}(s) + n^{p-1} f_{p,1}(s) + \dots. \tag{58}$$

To set up differential equations for the $f_{p,j}$ one has to compute

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle \mathscr{G}^p \rangle_s = \langle \tfrac{1}{2} \Delta \mathscr{G}^p \rangle_s. \tag{59}$$

For this a suitable choice of coordinates has to be made. We distinguish the three cases with an index $\varepsilon = 0, 1, -1$ for $SU(n,n)$, $Sp(n,\mathbb{R})$, $SO^*(2n)$. Without going into details, we remark that the embedding of $Sp(n,\mathbb{R})/U(n)$ and $SO^*(2n)/U(n)$ in $SU(n,n)/S(U(n)\times U(n))$ as complex manifolds allows to a great extent, simultaneous treatment of the different systems. After adequate rescaling of length in all cases, we are led to the following system of equations:

$$(n+\varepsilon)\frac{d}{ds}\langle \mathcal{G}^p\rangle_s = \langle -p\mathcal{G}^{p+1} - \varepsilon p \mathcal{G}_2\mathcal{G}^{p-1} - (1+\varepsilon^2)p(p-1)(\mathcal{G}_3 - \mathcal{G}_2)\mathcal{G}^{p-2}\rangle_s$$

$$(n+\varepsilon)\frac{d}{ds}\langle \mathcal{G}_2\mathcal{G}^{p-1}\rangle_s$$
$$= \Big\langle -(p+3)\mathcal{G}_2\mathcal{G}^p + 2\mathcal{G}^{p+1} + 2\varepsilon\Big(\mathcal{G}_2\mathcal{G}^{p-1} - 2\mathcal{G}_3\mathcal{G}^{p-1} - \frac{p-1}{2}\mathcal{G}_2^2\mathcal{G}^{p-2}\Big)\Big\rangle_s + O(n^{p+1}) \tag{60}$$

$$(n+\varepsilon)\frac{d}{ds}\langle \mathcal{G}_3\mathcal{G}^{p-1}\rangle_s = \langle -(p+5)\mathcal{G}_3\mathcal{G}^p + 6\mathcal{G}_2\mathcal{G}^p - 3\mathcal{G}_2^2\mathcal{G}^{p-1}\rangle_s + O(n^p)$$

$$(n+\varepsilon)\frac{d}{ds}\langle \mathcal{G}_2^2\mathcal{G}^{p-2}\rangle_s = \langle -(p+6)\mathcal{G}_2^2\mathcal{G}^{p-1} + 4\mathcal{G}_2\mathcal{G}^p\rangle_s + O(n^p)$$

(compare with [2] for the $\varepsilon = 1$ case). Here

$$\mathcal{G}_k^l(Z) = \mathrm{tr}((1^{(n)} - Z^+Z)^k)^l \qquad \mathcal{G}_1^l = \mathcal{G}^l. \tag{61}$$

while Z was introduced in the context of equation (15). Introducing asymptotic expansions with respect to the number of channels as well for the other functions $\langle \mathcal{G}_2\mathcal{G}^{p-1}\rangle_s, \langle \mathcal{G}_3\mathcal{G}^{p-1}\rangle_s, \langle \mathcal{G}_2^2\mathcal{G}^{p-2}\rangle_s$, one arrives at systems of ordinary differential equations which may be solved successively. As a variable, it is useful to introduce $x = s+1$ since everything has to be regular as $s \to 0$. The outcome is

$$\langle \mathcal{G}^p\rangle_s = n^p\frac{1}{(s+1)^p} - n^{p-1}\frac{\varepsilon p s^3}{3(s+1)^{p+2}} + n^{p-2}\frac{1}{(s+1)^{p+4}}\sum_{k=0}^{6}\gamma_k(p)(s+1)^k + O(n^{p-3}) \tag{62}$$

with

$$\begin{aligned}
\gamma_0(p) &= \frac{p}{18}[(4\varepsilon^2+3)p + (4\varepsilon^2-1)]\\
\gamma_1(p) &= -\frac{p}{15}[(3\varepsilon^2+8)p + (\tfrac{27}{2}\varepsilon^2 + \tfrac{5}{2})]\\
\gamma_2(p) &= \tfrac{1}{3}\varepsilon^2 p + \frac{3\varepsilon^2+2}{3}p(p+1)\\
\gamma_3(p) &= -\tfrac{1}{9}p[10\varepsilon^2 p - \tfrac{1}{2}(7\varepsilon^2-1)]\\
\gamma_4(p) &= \tfrac{5}{6}\varepsilon^2 p(p-1)\\
\gamma_5(p) &= -\tfrac{1}{3}\varepsilon^2 p(p-1)\\
\gamma_6(p) &= \tfrac{1}{90}p[(8\varepsilon^2+3)p - (4\varepsilon^2+5)].
\end{aligned} \tag{63}$$

Finally

$$\mathrm{var}(\mathcal{G})_\infty = \lim_{s\to\infty}\lim_{n\to\infty}(\langle \mathcal{G}^2\rangle_s - \langle \mathcal{G}\rangle_s^2) = \frac{1+\varepsilon^2}{15}. \tag{64}$$

This result agrees with Mello's original result of the $\varepsilon = 1$ case [2] when the spin degeneracy factor in his calculation is taken into account.

8. Summary

In the present work we offered an alternative view on a theory of quasi-one-dimensional conductors. The use of geometric notions and natural structures at hand allows some intuitive insight and a global setup of the model. In comparison, Mello used a maximum-entropy principle and constructed the model from infinitesimal considerations. As a further extension, the classification with respect to different kinds of time-reversal symmetry was completed and a list of the physical results may be given as follows:

Time reversal	Transfer matrices	Localization length	$\mathrm{var}(\mathcal{G})_\infty$	
none	$U(1)\times SU(n,n)$	$\sim 8n$	$\frac{1}{15}$	
integer spin	$Z_2\times Sp(n,\mathbb{R})$	$\sim 4(n+1)$	$\frac{2}{15}$	(65)
half-integer spin	$Z_2\times SO^*(2n)$	$\sim 8(n-1)$	$\frac{2}{15}$.	

As in related cases, the three universality classes are reminiscent of the classical types of Cartans list of simple complex Lie groups. The appealing feature of inducing the distribution function by an explicit average over possible but unobserved internal structures of a sample is accompanied by the drawback that still no information about the 'non-Abelian phases' K_0 is available. This seems to be a matter of additional microscopic input.

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